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Ioffe's Normal Cone and the Foundations of Welfare Economics

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Abstract. We report a version of the second fundamental theorem of welfare economics with Ioffe's normal cone as a formalization of marginal rates of substitution. Since Ioffe's normal cone is, in general, strictly contained in Clarke's normal cone, our results generalize earlier work of Khan-Vohra, Cornet and others.

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1. Introduction

In a strikingly modern treatment, Lange [19] showed in 1942 that at any Pareto optimal allocation of resources, the marginal rates of substitution in consumption and in production are identical; also see Hicks [13] and Allais [1]. Lange derived his result by observing that a Pareto optimal allocation of resources is a solution to a constrained maximization problem and hence the various marginal rates, in being equated to Lagrange multipliers corresponding to the material balance constraints, are necessarily equated to each other. Not surprisingly, Lange's primary hypotheses related to the differentiability of the utility and production functions.

Within a period of less than 10 years from Lange's work, Arrow [2] and Debreu [10] shifted the focus to the question of the decentralization of Pareto optimal allocations as price equilibria. They showed differentiability assumptions to be irrelevant to this question and emphasized instead convexity hypotheses on tastes and technologies with a corresponding emphasis on the separating hyperplane (Hahn-Banach) theorem. Their result, the so-called second fundamental theorem of welfare economics, stated that corresponding to any Pareto optimal allocation, there exist, under convexity, prices at which expenditure minimization by consumers and profit maximization by producers sustains the given allocation. If utility and production functions happened to be differentiable at the Pareto optimal consumption and production plans, this implied the equality of the marginal rates but these equalities themselves played no role in either the existence

of the decentralizing prices or in their principal properties. The definitive statement appeared in Debreu [11].

The Arrow-Debreu approach has been fundamental for subsequent work on the price characterization of Pareto optimal allocations. With its emphasis on the separation of convex sets, it was ideally suited for environments without utility functions, as in the case of consumers with non-ordered preferences; or in those without production functions, as in economies with infinite commodities. Ironically, it also had a profound influence on research whose principal object was to relax the convexity hypothesis. Thus, in his pioneering study on the price-decentralization of Pareto optimal allocations of non-convex economies, Guesnerie [14] assumed that the "local" shape of the sets at the Pareto optimal allocation was convex. This allowed him to "locally separate" the sets, and show that corresponding to any Pareto optimal allocation, there exist prices at which local expenditure minimization and local profit maximization sustains the given allocation; also see Otani-Sicilian [20].

A complete solution to Guenerie's problem had to await the discovery of a tangential approximant which was convex even if the set was not locally convex but which reduced to the local shape of the set if local convexity did obtain. Such an approximant is the Clarke [7] tangent cone and it was used by Khan-Vohra [17] (also see Quinzii [21] and Yun [25]) to show that corresponding to any Pareto optimal allocation, there exists a price system which is in the polar of the Clarke tangent cone, the Clarke normal cone, to the production sets and the "no-worse-than" sets at the Pareto optimal production and consumption

plans. Khan-Vohra assumed "desirability" and "free disposal" but made no local convexity assumptions of any kind. However, the point to be made is that the emphasis on convexity and the separation of convex sets remained in their work. Moreover, their result could be seen as a synthesis of previous work in the following sense. If the underlying sets were convex, as in Arrow-Debreu, the Clarke normal cone is identical to the normal cone in the sense of convex analysis and one obtains the characterization of Pareto optimal plans as profit maximizers or expenditure minimizers. If the underlying sets were locally convex at the Pareto optimal allocation, as in Guesnerie, the characterization involves local maximizing and minimizing behavior. If the sets were smooth in the sense of the Clarke normals being unique (of course up to a scalar multiple), one obtains the equality of marginal rates as in Lange.

This synthesis leads one to view the Clarke normal cones as a generalization of the concept of a marginal rate of substitution and argue for the primacy of Lange's viewpoint with the only modification being that at a Pareto optimal allocation these generalized marginal rates have a nonempty intersection. This implies equality when and only when these rates are singletons (again modulo scalar multiples). If convexity is present, we obtain, as a consequence, the characterization of these plans as maximizers or minimizers, but, to turn the matter on its head, this convexity plays no role in the existence of the multipliers themselves.

However, this alternative viewpoint hinges on how satisfactory one finds the Clarke normal cone as a formalization of marginal rates of

substitution. In a recent paper, Cornet [9] has generalized the results of Khan-Vohra to a situation where <u>none</u> of the technologies need exhibit "free disposal." Impressive as this generalization is on a technical level, it raises the question as to the adequacy of the Clarke normal cone as a formalization of a marginal rate of substitution in production without "free disposal." The point can be illustrated simply in terms of Figure 1 which depicts the Pareto optimal allocation of a single consumer, single producer economy. The technology is given by OAB and the no-worse-than set by the quadrant CAD. It is easy to check that the Clarke normal cone to the quadrant CAD at A is the quadrant EAF shifted to the origin while that to OAB at A is the whole space! Thus, as guaranteed by Cornet's result, the intersections of these normal cones has a nonempty intersection. However, this result is hardly sharp enough. Moreover, in terms of our intuitive notion of a marginal rate of substitution, marginal product in this case, to OAB at A, we would like to confine ourselves to the cone HAG and rule out, in particular, a vector such as p. This example brings out clearly the cost of requiring the tangential approximant to a non-covex set to be convex always. In the case of OAB, the best convex tangent cone that one can come up with consists of zero. An example similar to that presented in Figure 1 can also be found in Graaf [12, page 109].

Motivated by this example, one then looks for a tangential approximant which is sharper than that of Clarke and which accords more with our intuitive notion of a marginal rate of substitution. In

this paper we offer such an approximant. This is a normal cone introduced by Ioffe [15] and is based on his notion of an approximate subdifferential. The technical difficulty with the use of this cone lies in the fact that it is not necessarily convex. However, this difficulty can be surmounted if we stop thinking in terms of separation of sets and revert to programming theory as in Lange and, more recently, Cornet [9]; also see Debreu [10] and Yun [24].

It is important to be clear why, in the generality we work in, an approach based on separation of sets is very distinct from that of non-smooth programming. In the conventional theory of quasi-concave programming, as in Arrow-Enthoven [3], the basic results are, in any case, derived through the use of the separating hyperplane theorem. Thus, for the price decentralization of Pareto optimal allocations, whether one uses quasi-concave programming or applies the separating hyperplane theorem directly, is very much a matter of taste. For a non-convex, non-smooth set-up, the situation is completely different. The basic results in non-smooth programming theory are based on <u>surjection</u> theorems. These are results which offer sufficient conditions under which a correspondence maps a neighborhood of a point onto a neighborhood of the image of the point. We offer a brief description of this approach by considering the problem P:

$$\text{Maximize } f(x) \text{ subject to } 0 \in F(x),$$

where f is a function on X and F a correspondence from X to Y. Consider a mapping G: $X \to R \times Y$ where $G(x)$: $\{(\alpha, y)\colon \ \alpha \geq f(x) \ y \in F(x)\}$. If z is a local solution to P, then there does not exist any

neighborhood of x which is mapped by G into a set containing a neighborhood of $(f(z),0)$. Thus G is not surjective and hence we obtain a negation of the sufficient conditions under which it is necessarily surjective. It is this negation that furnishes the multipliers.

The outline of the remainder of this paper is as follows. In Section 2 we present a self-contained description of Ioffe's normal cone and relate it, in particular, to that of Clarke. In Section 3 we present Kuhn-Tucker type necessary conditions for a programming problem. These constitute the principal technical result underlying our work and they may have independent interest. Section 4 presents the economic model and results and Section 5 is devoted to the proofs. Section 6 concludes the paper with two remarks.

2. Ioffe's Normal Cone

Let R^n denote n-dimensional Euclidean space with R^n_+ as its non-negative orthant and with $\gg, >, \geq$ as the ordering on vectors. For any set $Y \subset R^n$, $C\ell A$, con A, Int A will be used to denote its closure, convex hull and interior respectively. $B_\varepsilon(y)$ $(\overline{B}_\varepsilon(y))$ denotes the open (closed) ball with center y and radius ε. $[a,b)$ will denote the set $\{x \varepsilon R: a \leq x < b\}$.

We begin by recalling the definition of the Clarke tangent cone which was first introduced into the economic literature by Cornet [8].

Definition 2.1. The Clarke tangent cone of $Y \subset R^n$ at $y \varepsilon Y$, $T_C(Y,y)$, consists of the set $\{x \varepsilon R^n$: For any sequence (t^k, y^k) in $R_+ xY$ with $t^k > 0$ and tending to $(0,y)$, there exists a sequence (x^k) tending x such that $(y^k + t^k x^k) \varepsilon Y$ for large enough $k\}$.

For an alternative equivalent definition and discussion, the reader can see, for example, [17]. He can also check that the Clarke tangent cone to the technologies OAB at A in Figure 2 consists solely of the zero element. The Clarke tangent cone at 0 to the technology depicted in Figure 3, on the other hand, is the negative quadrant bounded by OC and OD.

Recall that the Clarke normal cone is the <u>polar cone</u> to the Clarke tangent cone, i.e., the set $N_C(Y,y) = \{x \in R^n: \ (x,z) \leq 0$ for all $z \in T_C(Y,y)\}$. It is evident from the examples in Figure 2 that the Clarke tangent cone at A to OAB is "too small" and that the normal cone is "unsatisfactorily large."

Next we turn to a tangential approximant introduced by Bouligand [5] and termed the contingent cone. It was first applied in economics by Otani-Sicilian [20].

<u>Definition 2.2</u>. The <u>contingent cone</u> of $Y \subset R^n$ at $y \in Y$, $T_K(Y,y)$, consists of the set $\{x \in R^n$: There exists a sequence (t^k) in R_+ with $t^k > 0$ and tending to zero and a sequence (x^k) tending to x such that $(y+t^k x^k) \in Y$ for large enough $k\}$.

The contingent cone is obtained if, in terms of the definition of the Clarke tangent cone, one limits oneself to the constant sequences (y^k) with y^k equal to y. As such, it is larger than the Clarke tangent cone. We leave it to the reader to check that the normal contingent cone to OAB at A in Figures 2a and 2b is given, respectively, by the shaded cones shifted to the origin. It can also be checked

that the contingent cone at 0 to the technology of Figure 3 is the cone enclosed by AO and OD.

It is a simple matter to show that the second welfare theorem no longer obtains with the formalization of marginal rates of substitution as the normals to the contingent. Consider the Pareto optimal allocation of the economy described in Figure 1. The contingent normals to OAB consist of the cone enclosed by KAL and shifted to the origin while that to the set enclosed by CAD is the negative orthant. Thus there exists no price system which is in the normal cone to OAB at A but whose negative is in the negative orthant. For another example, see Bonnisseau-Cornet [4].

The example in Figure 1 clearly brings out the fact that, if the Clarke tangent cone is "too small," the contingent cone is "too large." To put the matter in terms of the normal cones, the Clarke normal cone is "too large" and the normal contingent cone is "too small." If this normal cone could somehow be enlarged to include the vectors AH and AG shifted to the origin, we would obtain an extension of the second welfare theorem that is not only "sharper" than the one based on the Clarke normal cone but also more in keeping with our intuitive ideas of marginal rates of substitution.

This "enlargement" of the normal contingent cone is precisely what is accomplished by the normal cone introduced by Ioffe. The basic idea is an obvious one. It is to include in the definition of a normal cone not only all the vectors in the contingent normal cone but also the limits of vectors in the contingent normal cones at "nearby" points. In terms of Figure 1, consider all production plans "near" A

on the segment AB. The contingent normal cone to OAB at such plans is given by the perpendicular to AB shifted to the origin. Similarly, the contingent normal cone to OAB at all production plans "near" A on the segment OA is given by the perpendicular to OA shifted to the origin. Thus the Ioffe normal cone to OAB at A is the cone enclosed by KAL and the vectors AH and AG, all shifted to the origin. It is an analytical object which is precisely what we are looking for.

For a formal definition, we first recall that the lim sup (A^k) of a sequence of sets $A^k \subset R^n$, consists of cluster points of sequences chosen from A^k. Thus, $\limsup A^k = \{x \in R^n : \exists \text{ a subsequence } (x_{k_i}),\ x_{k_i} \in A^{k_i} \text{ such that } x_{k_i} \to x\}$. We can now present

Definition 2.3. The Ioffe normal cone, $N_a(Y,y)$, to $Y \subset R^n$ at $y \in Y$ is given by $\limsup\limits_{\substack{z \to y \\ z \in Y}} N_K(Y,z)$.

The reader can check his understanding of the Ioffe normal cone by verifying that it is given to OAB at A in Figure 2a by the shaded cone and the vectors AK and AL, all shifted to the origin at A. The Ioffe normal cone at 0 to the technology in Figure 3 is given by the non-negative orthant.

In all of our examples, the Ioffe normal cone contains the contingent normal cone and is contained in the Clarke normal cone. This containment need not be strict, of course, as in Figure 3. Moreover, the reader may also have observed that in all of our examples the Clarke normal cone is the closed convex hull of the Ioffe normal cone.

These are general results but before noting them as a formal proposition, we present an alternative motivation of the Ioffe normal cone.

The idea of a normal to a set Y at a point y in Y embodies in it the idea of vectors to which the "closest" point in Y is y. Note that there are no normal vectors in this sense to Y at y in Figure 3. However, in Figures 2a and 2b, the shaded cones constitute such normals to OAB at A. On shifting such a set of normals to the origin, we obtain the <u>proximant normal cone</u>. More formally, we can state

<u>Definition 2.4.</u> The <u>proximant normal cone</u>, $N_P(Y,y)$, to $Y \subset R^n$ at $y \in Y$ is given by $\{x \in R^n\colon \exists\, \varepsilon > 0$ such that for all $t \in (0,\varepsilon)$,

$$\|(tx+y)-y\| = \inf_{z \in Y} \|(tx+y)-z\|\}.$$

We can now state

<u>Proposition 2.1.</u> For any closed set $Y \subset R^n$ and for any $y \in Y$, the following is true.

(i) $\quad N_C(Y,y) = C\ell \text{ con } N_a(Y,y)$

(ii) $\quad N_a(Y,y) = \limsup_{\substack{z \to y \\ z \in Y}} N_P(Y,z)$

<u>Proof.</u> See the proofs of Theorems 1 and 2 in Ioffe [16]. ||

We conclude this section by referring the reader to Figure 4. It is easy to check that the Clark normal cone to Y at A is given by the shaded cone HAG shifted to the origin, the Ioffe normal cone is the set solely consisting of the vectors AH and AG shifted to the origin, while the contingent normal cone is the origin itself.

3. A Result on Programming

In this section we present a result on mathematical programming that is the principal technical result used in this paper and that may have an independent interest.

Consider the following standard problem of mathematical programming which we shall denote as Problem P and in which f_i are real-valued functions.

Minimize $f_0(x)$

$$\begin{aligned} \text{subject to } f_i(x) &\leq 0 \qquad i = 1, \ldots, m \\ f_i(x) &= 0 \qquad i = m+1, \ldots, \ell. \\ x &\in S \subset R^n \end{aligned}$$

Before we write down the necessary conditions corresponding to Problem P, we develop the notion of an "Ioffe subdifferential" of a nondifferentiable function. For this, we need only recall the definition of an epigraph of a function. Formally, for any f: $R^n \to R^1$,

$$\text{epif} = \{(\alpha, x) \in R^1 \times R^n: \ \alpha \geq f(x)\},$$

i.e., the set of all points on or above the graph of f. The Ioffe subdifferential of f at a point z at which $|f(z)|$ is finite, $\partial_a f(z)$, is given by the set

$$\{x^* \in R^n: \ (-1, x^*) \in N_a(\text{epif}, (f(z), z))\}.$$

The following properties of the Ioffe subdifferential are of interest.

Proposition 3.1. Let $f: R^n \to R$

(i) if f is strictly differentiable at z, that is, there is an x^* such that

$$\limsup_{\substack{h \to 0 \\ x \to z}} \|h\|^{-1} | f(x+h)-f(x) - \langle x^*,h\rangle | = 0,$$

then $\partial_a f(z) = \{x^*\}$.

(ii) If f is a convex function, then

$$\partial_a f(z) = \{x^*: \ f(x)-f(z) \geq \langle x^*,u-z\rangle \text{ for all } x\}$$

is a subdifferential of f at z in the sense of convex analysis.

Proof. See Propositions 4, 5 and 2 in Ioffe [16]. ||

We can now present

Theorem 3.1 (Ioffe). Let z be a local solution to Problem P. If f_i, $i=0, \ldots, n$ are Lipschitz near z and S is a closed set, there exist Lagrange multipliers λ_i, $i=0, \ldots, n$, not all zero, such that

(i) $$0 \in \partial_a\left(\sum_{i=0}^{n} \lambda_i f_i\right)(z) + N_a(S,z)$$

(ii) $$\lambda_i \geq 0 \text{ and } \lambda_i f_i(z) = 0, \ i=1, \ldots, m.$$

Proof. See the proof of Proposition 12 in Ioffe [16]. ||

We may note in passing that Ioffe's proof of Theorem 3.1 is based on a surjection theorem as described in the introduction above. We may also note that Theorem 3.1 requires no constraint qualification

for its validity. This is simply because we include a multiplier for the objective function and do not guarantee that this multiplier is non-zero. Without such a guarantee, no constraint qualification is needed even in the traditional Kuhn-Tucker theory; see Uzawa's [24] original statement.

4. The Model and Results

We consider an economy with n commodities, T consumers and F firms. We shall index consumers by t, t=1, ..., T, each having a consumption set $X^t \subset R^n_+$ and preferences described by the sets $P^t(x^t)$, $x^t \in X^t$, with the interpretation that these are the sets of commodity bundles preferred by t to the bundle x^t. Firms are indexed by j, j=1, ..., F, each having a production set $Y^j \subset R^n$. The aggregate endowment is denoted by $w \in R^n_+$. An economy is thus denoted by $\mathcal{E} = ((X^t, P^t(\cdot))_1^T, (Y^j)_1^F, w)$. We shall need the following concepts for $\mathcal{E}$.

Definition 4.1. $((x^{*t})(y^{*j}))$ is an allocation of $\mathcal{E}$ if for all t=1, ..., T, $x^{*t} \in X^t$, for all j=1, ..., F, $y^{*j} \in Y^j$ and $\sum_t x^{*t} - \sum_j y^{*j} = w$.

Definition 4.2. An allocation $((x^{*t}), (y^{*j}))$ is Pareto optimal if there does not exist any other allocation $((x^t), (y^j))$ such that $x^t \in C\ell\ P^t(x^{*t})$ for all t and $x^t \in P^t(x^{*t})$ for at least one t.

We now present a basic assumption on a given allocation of .
This is taken from Cornet [9].

Definition 4.3. An allocation $((x^{*t}), (y^{*j}))$ satisfies Cornet's constraint qualification if there exist $\tau \in \{1, \ldots, T\}$, $e \in R^\ell$ and $\delta > 0$ such that for all λ in $(0,\delta)$,

$$(*)\qquad \lambda e + \sum_t (C\ell\ P^t(x^{*t}) \cap B_\delta(x^{*t})) - \sum_j (C\ell Y^j \cap B_\delta(y^{*j})) \subset P^\tau(x^{*\tau}) + \sum_{t\neq i} C\ell\ P^t(x^{*t}) - \sum_j Y^j$$

We can now present

Theorem 4.1. If $((x^{*t}), (y^{*j}))$ is a Pareto optimal allocation satisfying Cornet's constraint qualification, there exists $p^* \varepsilon R^n$, $p^* \neq 0$ such that

(a) $\quad -p^* \varepsilon N_a (C\ell P^t(x^{*t}), x^{*t})$ for all t,

(b) $\quad p^* \varepsilon N_a(C\ell Y^j, y^{*j})$ for all j.

The following proposition is taken from Cornet [9].

Proposition 4.1 (Cornet). Cornet's constraint qualification is satisfied at $((x^t), (y^j))$ if

(i) $P^t(x^t)$ is convex for all t and $\sum_j Y^j$ is convex, or if

(ii) there exist $e \varepsilon R^\ell$, and $\delta > 0$ such that for all $\lambda \varepsilon (0,\delta)$, $\lambda e + (C\ell C \cap B_\delta(c)) \subset \text{Int } C$ for (C,c) equal to any one of $(-Y^j,-y^j)$ and $(P^t(x^t), x^t)$, or if

(iii) Y^j is closed or convex for all j and there exists $e \varepsilon R^\ell$ and $\delta > 0$ such that for all $\lambda \varepsilon (0, \ )$, $\lambda e + (C\ell C \cap B_\delta(c)) \subset C$ for (C,c) equal to any one of $(P^t(x^t),x^t)$.

Proof. See the proofs of Propositions 3.2 and 3.5 in Cornet [9].

Armed with Propositions 4.1 and 2.1, we can see that our Theorem 4.1 generalizes all the results in Cornet [9], which in turn

generalized the earlier results of Arrow [2], Debreu [9, 10], Guesnerie [14], Khan-Vohra [17] and others in a finite commodity, private goods setting. All that needs to be checked is whether $-p^* \in T_C(P^t(x^{*t}),x^{*t})$ for all t, and $p^* \in T_C(Y,y^{*j})$ for all j. From Proposition 2.1, $-p^* \in N_a(C\ell P^t(x^{*t}),x^{*t})$ implies $-p^* \in N_C(C\ell P^t(x^{*t}),x^{*t})$. However, it is well known and easily verified from the definition that the latter set equals $N_C(P^t(x^{*t}),x^{*t})$ since $x^{*t} \in C\ell P^t(x^{*t})$. The verification for production sets follows an identical argument.

At this stage a natural question arises as to whether the closure operator can be omitted from (a) and (b) in the statement of Theorem 4.1. The fact that this is not possible without additional hypotheses is illustrated in Figure 5 which depicts a single producer, single consumer economy in which the initial endowment and the Pareto optimal allocations are given by the origin. We leave it to the reader to check that Cornet's constraint qualification is satisfied for this economy at ((0),(0)) and that the Ioffe normal cone to Y at zero consists solely of the zero vector. Hence there does not exist any $p^* \neq 0$ which satisfies the conclusion of Theorem 4.1.

It is of some interest that the closure operator can be omitted for those of the sets $P^t(x^{*t})$ or Y^j that are convex or satisfy a weak closedness condition. We formalize this next.

<u>Definition 4.4.</u> $C \subset R^n$ is said to be <u>tangentially closed</u> at $c \in C$ if for all $x \in N_a(C\ell C,c)$, there exists a sequence $\{c^k\}$ in C, c^k tending to C, and $x^k \in N_K(C\ell C,c^k)$ such that x^k tends to x.

Note that it is the requirement that c^k belongs to C, rather than $C\ell C$, that makes the above definition interesting. Thus, it is obvious that every closed set is tangentially closed at each of its points. It is also trivial to see that if $C \subset \overline{B}_\varepsilon(c)$ is closed for any $\varepsilon > 0$, C is tangentially closed at C. Convex sets possess points at which they are not tangentially closed. In Figure 3, the cone enclosed by OB and OD, including O but not any other points on the lines OB and OD, is convex but not tangentially closed at O. Moreover, the set Y in Figure 3 with all points of its boundary, except for those lying on OA, deleted, is tangentially closed at O but not a closed set even within an arbitrarily small neighborhood of O.

We can now state

Corollary 4.1. If, in Theorem 4.1, $P^t(x^{*t})$ is convex or tangentially closed at x^{*t}, then (a) can be rewritten as $-p^* \varepsilon N_a(P^t(x^{*t})\{x^{*t}\},x^{*t})$. If Y^j is convex or tangentially closed at y^{*j}, then (b) can be rewritten as $p^* \varepsilon N_a(Y^j,y^j)$.

Next, we turn our attention to public goods. We work in R^{n+m} where the first n commodities are private goods and the next m are public goods in the Samuelsonian [23] sense that their consumption is identical across consumers. An economy with public goods $\mathcal{E}^G = ((X^t, P^t(\cdot))_1^T, (Y^j)_1^F, w)$ is such that for all t, $X^t = (X_\pi^t, X_g^t)$ where $X_\pi^t \subset R_+^n$, $X_g \subset R_+^m$ are the projections of X^t onto the space of private and public goods, respectively. We assume that $X_g^t = X_g$ for all t; that $Y^j \subset R^{n+m}$ for all j; that $w = (w_\pi, 0)$, $w_\pi \in R_+^n$; and we let $x_{\pi\ell}^t$ and $x_{g\ell}^t$

denote the consumption of t of the ℓth private and ℓth public good, respectively.

Definition 4.5. $((x^{*t}), (y^{*j}))$ is an allocation of $\mathcal{E}^G$ if for all $t=1, \ldots, T$, $x^{*t} \in X^t$, for all $j=1, \ldots, F$, $y^{*j} \in Y^j$, $x_g^{*t} = x_g^*$ for all t, and

$$(\sum_t x_\pi^{*t}, x_g^*) - \sum_j y^{*j} = w.$$

The definition of a Pareto optimal allocation for $\mathcal{E}^G$ is identical to the one given in Definition 4.2.

In an economy with public goods, it is natural to assume that there is at least one individual who desires the public good. Towards this end, we develop

Definition 4.6. An allocation $((x^{*t}), (y^{*j}))$ is non-satiated in public goods if there exist $\tau \in \{1, \ldots, T\}$, $e_g \in R^m$ and $\varepsilon > 0$ such that for all $\lambda \in (0,\varepsilon)$,

$$(**) \qquad \lambda(0,e_g) + (C\ell P^\tau(x^{*\tau}) \cap B_\varepsilon(x^{*\tau})) \subset P^\tau(x^{*\tau})$$

We can now state

Theorem 4.2. If $((x^{*t}, y^{*j}))$ is a Pareto optimal allocation of $\mathcal{E}^G$ which is non-satiated in public goods, and Y^j is closed for all j, then there exists $p^* = (p_\pi^*, p_g^*) \in R^{n+m}$ and, for all t, $p_g^{*t} \in R^m$ with $(p_\pi^*, (p^{*t})) \neq 0$ and such that

(i) $\sum_t p_g^{*t} = p_g^*$,

(ii) $-(p_\pi^*, p_g^{*t}) \in N_a(C\ell P^t(x^{*t}), x^{*t})$ for all t,

(iii) $p^* \in N_a(Y^j, y^{*j})$ for all j.

Theorem 4.2 should be compared to Theorem 2 in Khan-Vohra [17]. They work with the Clarke normal cone and assume "free disposal" for all production sets and a weak form of desirability of public goods for all individuals. However, they do not require the production sets to be closed. It seems reasonable to conjecture that a version of our Theorem 4.2 could be proved without the closedness assumption but the complicated form of a constraint qualification that will be undoubtedly needed seems hardly worth the effort.

5. Proofs

We begin this section by developing four properties of the Ioffe normal cone.

Lemma 5.1. Let $Y^i \subset R^{n_i}$ for i between 1 and ℓ, $Y = \prod_i Y^i$ and $y = (y_1, \ldots, y_\ell) \in Y$ with $y_i \in Y^i$. Then

$$N_a(Y,y) = \prod_i N_a(Y^i, y_i)$$

Lemma 5.2. Let Y^1 and Y^2 be closed subsets of R^n such that $z \in Y^1 \cap Y^2$ and $T_C(Y^1,z) - T_C(Y^2,z) = R^n$. Then

$$N_a(Y^1 \cap Y^2, z) \subset N_a(Y^1,z) + N_a(Y^2,z)$$

Lemma 5.3. For any $Y \subset R^n$, let $y \in \text{Int } Y$. Then $N_a(Y,y) = \{0\}$.

Lemma 5.4. If $Y \subset R^n$ is convex or tangentially closed at $y \in Y$, then

$$N_a(Y,y) = N_a(C\ell Y, y).$$

Note that the conclusion of Lemma 5.4 is false, in general, for a non-convex set Y that is not tangentially closed at y. To see this, let Y be as in Figure 5 with $y = 0$. Then $N_a(C\ell Y,0) \neq N_a(Y,0) = \{0\}$.

We begin with a

Proof of Lemma 5.1. We shall prove Lemma 5.1 for the case $\ell=2$; the general case then follows easily by induction.

We first observe that

$$T_K(Y,y) = T_K(Y^1,y_1) x T_K(Y^2,y_2).$$

To see this pick any $x = (x_1,x_2) \in T_K(Y,y)$ with x_1 and x_2 corresponding to the n_1 and n_2 coordinates respectively. By definition of the contingent cone, there exists a sequence (t^k,x^k) tending to $(0,x)$ with $t^k > 0$ and such that $(y+t^kx^k) \in Y$ for all large enough k. But $(y+t^kx^k) \in Y$ implies $(y_i+t^kx_i^k) \in Y^i$, $i=1,2$, and we have shown that $x_i \in T_K(Y^i,y_i)$, $i=1,2$. This implies

$$N_K(Y,y) \subset N_K(Y^1,y_1) \text{ x } N_K(Y^2,y_2).$$

To show the converse, we observe easily from the definitions that

$$T_K(Y,y) \subset T_K(Y_1,y_1) \times \{0\}$$

$$\subset \{0\} \times T_K(Y_2,y_2).$$

Now suppose $(x_1,x_2) \in N_K(Y,y)$ but $x_1 \notin N_K(Y_2,y_1)$. Then there exists $p \in T_K(Y_1,y_1)$ such that $\langle p,x_1\rangle > 0$. But, since $(p,0) \in T_K(Y,y)$, we obtain a contradiction.

Given Definition 2.3, all that remains is to show that

$$\limsup_{\substack{z \to y \\ z\in Y}} N_K(Y,z) = \limsup_{\substack{z \to_1 y_1 \\ z\in Y^1}} N_K(Y^1,z) \times \limsup_{\substack{z \to_2 y_2 \\ z\in Y^2}} N_K(Y^2,z)$$

To see this, pick any x in the left hand set. Then there exists a sequence (z^k) chosen from Y and $x^k \in N_k(Y,z^k)$ such that $x^k \to x$ and $z^k \to y$. But this implies from the above argument that x is an element in the right hand side set. The converse is similar.

||

<u>Proof of Lemma 5.2.</u> Lemma 5.2 is Corollary 4.2 in Ioffe [16] and its proof is given there.

||

<u>Proof of Lemma 5.3.</u> Since

$$N_a(Y,y) = \limsup_{\substack{z \to y \\ z\in Y}} N_K(Y,z),$$

and since $y \in \text{Int } Y$, it suffices to show that $N_K(Y,z) = \{0\}$ for any interior point z of Y. We thus need to show that $T_K(Y,z) = R^n$. To see this, pick any sequence t^k of positive numbers tending to zero. For any $x \in R^n$, $(z+t^k x) \in Y$ since z is an interior point of Y, and hence $x \in T_K(Y,z)$.

||

For the proof of our last lemma, we need the following elementary results which are well known but for which we could find no detailed proofs.

<u>Lemma 5.5.</u> For any family of sets $\{A_i\}_{i\in I}$, $C\ell(\bigcup_{i\in I} A_i) = C\ell(\bigcup_{i\in I} C\ell A_i)$.

<u>Proof of Lemma 5.5.</u> Let $\overline{C} = \{F\subset R^n\colon\ F \text{ closed and } (\bigcup_{i\in I} C\ell A_i)\subset F\}$ and $C = \{F\subset R^n\colon\ F \text{ closed and } (\bigcup_{i\in I} A_i)\subset F\}$. The fact that $\overline{C}\subset C$ is clear. To see $C\subset\overline{C}$, pick $F \in C$. Then $A_i \subset F$ for all $i \in I$. Since F is closed, $C\ell A_i\subset F$ for all $i \in I$. But then $F \in \overline{C}$ and

$$C\ell\ (\bigcup_{i\in I} A_i) = \bigcap_{F\in C} F = \bigcap_{F\in \overline{C}} F = C\ell(\bigcup_{i\in I} C\ell A_i).$$

||

<u>Lemma 5.6.</u> For any $Y\subset R^n$ and $y \in Y$, $N_k(C\ell Y,y) = N_K(Y,y)$.

<u>Proof of Lemma 5.6.</u> We know (see, for example, [16, pg. 391]), that $N_K(Y,y) = \bigcap_{t>0} C\ell(\bigcup_{0<\gamma\leq t} \gamma^{-1}(Y-y))$. By Lemma 5.5, the formula on the right can be rewritten as

$$\bigcap_{t>0} C\ell(\bigcup_{0<\gamma\leq t} \gamma^{-1}(C\ell Y-y))$$

which in turn equals $N_K(C\ell Y,y)$. ||

<u>Proof of Lemma 5.4.</u> We first show, without any hypotheses on Y, that $N_a(Y,y)\subset N_a(C\ell Y,y)$. Pick $x \in N_a(Y,y)$. Then there exists a sequence $\{y^k\}$ in Y such that y^k tends to y and $x^k \in N_K(Y,y^k)$ with x^k tending to x. By Lemma 5.6, $x^k \in N_k(C\ell Y,y^k)$. Hence $x \in N_a(C\ell Y,y)$.

For the converse, let $x \in N_a(C\ell Y,y)$. By tangential closedness of Y at y, there exists a sequence $\{y^k\}$ in Y such that y^k tends to y and $x^k \in N_K(C\ell Y,y^k)$ with x^k tending to x. By another appeal to Lemma 5.6, $x^k \in N_K(Y,y^k)$ and we conclude $x \in N_a(Y,y)$.

Next, we consider the case when Y is convex. By considering the constant sequence $\{y\}$, it is clear that $N_K(Y,y) \subset N_a(Y,y)$. In the case that Y is convex, $N_K(Y,y) = N_C(Y,y)$ since both are equal to the closed normal cone to Y in the sense of convex analysis; see, for example, Theorem 1 in Rockafellar [22]. Hence $N_C(Y,y) \subset N_a(Y,y)$, the latter set, by Proposition 2.1, is in its turn contained in $N_C(Y,y)$. We are done. ||

Next, we turn to the proofs of our theorems. These are directly inspired by ideas of Cornet [9] which, in turn, can be traced to Debreu [10]; also see Yun [25]. What needs to be emphasized, as already done in the introduction, is that the essential ideas of the proofs are different from those of Khan-Vohra [17,18] or of Bonnisseau-Cornet [4].

Proof of Theorem 4.1.

Since $((x^{*t}),(y^{*j}))$ satisfies Cornet's constraint qualification, we know that there exist $\tau \in \{1, \ldots, T\}$, $e \in R^\ell$ and $\delta > 0$ such that (*) in Definition 4.3 is satisfied. Now, using e and ε with $0 < \varepsilon < \delta$, consider the following programming problem.

Maximize λ

subject to

(i) $\sum_t x^t - \sum_j y^j - w + \lambda e = 0$

(ii) $x^t \in C\ell P^t(x^{*t}) \cap \overline{B}_\varepsilon(x^{*t}) \quad t=1, \ldots, T$

(iii) $y^j \in C\ell Y^j \cap \overline{B}_\varepsilon(y^{*j}) \quad j=1, \ldots, F$

(iv) $\lambda \in [-\varepsilon, \varepsilon]$

We assert that $(0,(x^{*t}),(y^{*j}))$ is a local solution to the above programming problem. If not, we can find $(\overline{\lambda}, (\overline{x}^t), (\overline{y}^j))$ satisfying (i) to (iv) and with $0 < \overline{\lambda} < \delta$. By (*) in Definition 4.3, we know that there exist $((\hat{x}^t), (\hat{y}^j))$ such that $\hat{x}^\tau \in P^\tau(x^{*\tau})$, $x^t \in C\ell P^t(x^{*t})$, $t \neq \tau$, $\hat{y}^j \in Y^j$ for all j such that

$$\overline{\lambda} e + \sum_t x^{*t} - \sum_j y^{*j} = \sum_t \hat{x}^t - \sum_j \hat{y}^j$$

However, this contradicts the Pareto optimality of the allocation $((x^{*t}), (y^{*j}))$.

Now let

$$S = [-\varepsilon,\varepsilon] \times \prod_t (C\ell P^t(x^{*t}) \cap \overline{B}_\varepsilon(x^{*t})) \times \prod_j (C\ell Y^j \cap \overline{B}_\varepsilon(y^{*j})).$$

It is clear that S is a closed set and that the constraints (i), being linear, are differentiable. We can thus appeal to Theorem 3.1 due to Ioffe and assert the existence of $(\mu,p^*) \in R^{n+1}$, $\mu \in R$, $(\mu,p^*) \neq 0$ such that

$$0 \in \partial_a(\mu\lambda + \langle p^*, (\sum_t x^t - \sum_j y^j - w - \lambda e)\rangle)\,(0,(x^{*t}), (y^{*j}))$$

$$+ N_a(S,(0,(x^{*t})(y^{*j})).$$

From Proposition 3.1, we know that the Ioffe subdifferential coincides with the derivative and hence

$$-(\mu, \langle p^*, e\rangle, \prod_t p^*, \prod_j(-p^*)) \in N_a(S,(0,(x^{*t}),(y^{*j})).$$

By Lemma 5.1, we obtain

$$N_a(S,(0,(x^{*t}),(y^{*j}))) = N_a([-\varepsilon,\varepsilon],0)$$

$$\times \prod_t N_a((C\ell P^t\ (x^{*t}) \cap \overline{B}_\varepsilon(x^{*t})),x^{*t}) \times \prod_j N_a((Y^j \cap \overline{B}_\varepsilon(y^{*j})),y^{*j}).$$

Since 0 is in the interior of $[-\varepsilon,\varepsilon]$, the first cone on the right hand side is $\{0\}$. We thus obtain

(1) $\quad \langle p^*,e\rangle - \mu = 0,$

(2) $\quad -p^* \in N_a(C\ell P^t(x^{*t}) \cap \overline{B}_\varepsilon(x^{*t}),\ x^{*t}) \qquad t=1, \ldots, T,$

(3) $\quad p^* \in N_a(C\ell Y \cap \overline{B}_\varepsilon(y^{*j}),\ y^{*j}) \qquad j=1, \ldots, F.$

From (1), we can conclude that $p^* \neq 0$. If not, then $\mu = 0$ and we contradict the fact that $(\mu, p^*) \neq 0$.

It is well known and easy to check directly that $T_C(Y,y) = R^n$ for any interior point y of a set Y in R^n. We can thus appeal to Lemma 5.2 to rewrite (2) and (3) as

(4) $\quad -p^* \in N_a(C\ell P^t(x^{*t}),x^{*t}) + N_a(\overline{B}_\varepsilon(x^{*t}),x^{*t}) \qquad t=1, \ldots, T,$

(5) $\quad p \in N_a(C\ell Y^j, y^{*j}) + N_a(\overline{B}_\varepsilon(y^{*j}), y^{*j}) \qquad j=1, \ldots, F.$

Finally, an appeal to Lemma 5.3, along with the fact that $x^{*t} \in C\ell P^t(x^{*t})$ for all t, completes the proof. ‖

Proof of Corollary 4.1. Follows directly from Theorem 4.1 and Lemma 5.4. ‖

Proof of Theorem 4.2

Since $((x^{*t}), (y^{*j}))$ is non-satiated in public goods, we know that there exists $\tau \in \{1, \ldots, T\}$, $e_g \in R^m$ and $\delta > 0$ such that (**) in Definition 4.6 is satisfied. Now, using e_g and ε with $0 < \varepsilon < \delta$, consider the following programming problem.

Maximize λ

subject to

(i) $\quad \sum_t x^t_\pi - \sum_j y^j_\pi - w = 0$

(ii) $\quad x^t \in C\ell P^t(x^{*t}) \qquad t=1, \ldots, T, \; t\neq\tau$

(iii) $\quad x^\tau \in C\ell P^\tau(x^{*\tau}) \cap \overline{B}_\varepsilon(x^{*\tau})$

(iv) $\quad y^j \in Y^j \qquad j=1, \ldots, F$

(v) $\quad x^t_g = \sum_j y^j_g \qquad t=1, \ldots, T, \; t\neq\tau$

(vi) $\quad x^\tau_g + \lambda e_g = \sum_j y^j_g$

(vii) $\quad \lambda \in [-\varepsilon, \varepsilon].$

We assert that $(0,(x^{*t}),(y^{*j}))$ is a local solution to the above programming problem. If not, we can find $(\bar{\lambda},(\bar{x}^t),(\bar{y}^j))$ satisfying (i) to (vii) and with $0 < \bar{\lambda} < \varepsilon$. By (**) in Definition 4.5, we know that

$$\hat{x}^\tau \equiv \bar{\lambda}(0,e_g) + \bar{x}^\tau \in P^\tau(x^{*\tau})$$

Now let $\hat{x}^t = \hat{x}^t$ for all $t \neq \tau$. Then it is clear that the allocation $((\hat{x}^t)(\bar{y}^j))$ contradicts the Pareto optimality of $((x^{*t}),(y^{*j}))$.

Now, as in the proof of Theorem 4.1, we can appeal to Theorem 3.1 due to Ioffe and assert the existence of multipliers $(\mu, p^*_\pi,(p^{*t}_g)) \in R \times R^\ell \times R^{mT}$, not all zero, such that

(1) $\quad \langle p^{*\tau}_g, e_g\rangle + \mu = 0$

(2) $\quad -(p^*_\pi, p^{*t}_g) \in N_a(C\ell P^t(x^{*t}), x^{*t}) \qquad t=1, \ldots, T$

(3) $\quad (p^*_\pi, \sum_t p^{*t}_g) \in N_a(Y^j, y^j) \qquad j=1, \ldots, F$

We can claim that $(p^*_\pi,(p^{*t}_g)) \neq 0$. If not, then from (1) $\mu=0$ and we contradict the fact that $(\mu,p^*_\pi,(p^{*t}_g)) \neq 0$. Again, as in the proof of Theorem 4.1, we can appeal to Lemmata 5.2 and 5.3 to complete the proof. ||

6. Concluding Remarks

The programming result presented as Theorem 3.1 is as sharp as the notion of the normal cone and the subdifferential on which it is based; the smaller these sets, the better the result. An analogous statement can be made in the context of our Theorems 4.1 and 4.2 which

do not utilize the concept of a subdifferential but are based on the normal cone. Thus, a natural question arises as to whether the Ioffe normal cone is minimal among all cones satisfying properties which are needed in the proof of our welfare theorems. We leave this as an open question but refer the reader to Theorem 9 in [16] which shows that among the class of subdifferentials to a given class of functions, the Ioffe subdifferential is the smallest if the subdifferentials are required to satisfy four natural properties.

Our second remark relates to the non-convexity of the Ioffe normal cone. Whereas we have shown this property to be irrelevant for the second welfare theorem, it is important to bear in mind that the convexity of the Clarke tangent cone is essential in the proof of the existence of marginal cost pricing equilibria; see, for example, Brown et al. [6]. Whether one can show the existence, or the nonexistence, of a marginal cost pricing equilibrium based on the Ioffe normal cone seems an interesting open question.

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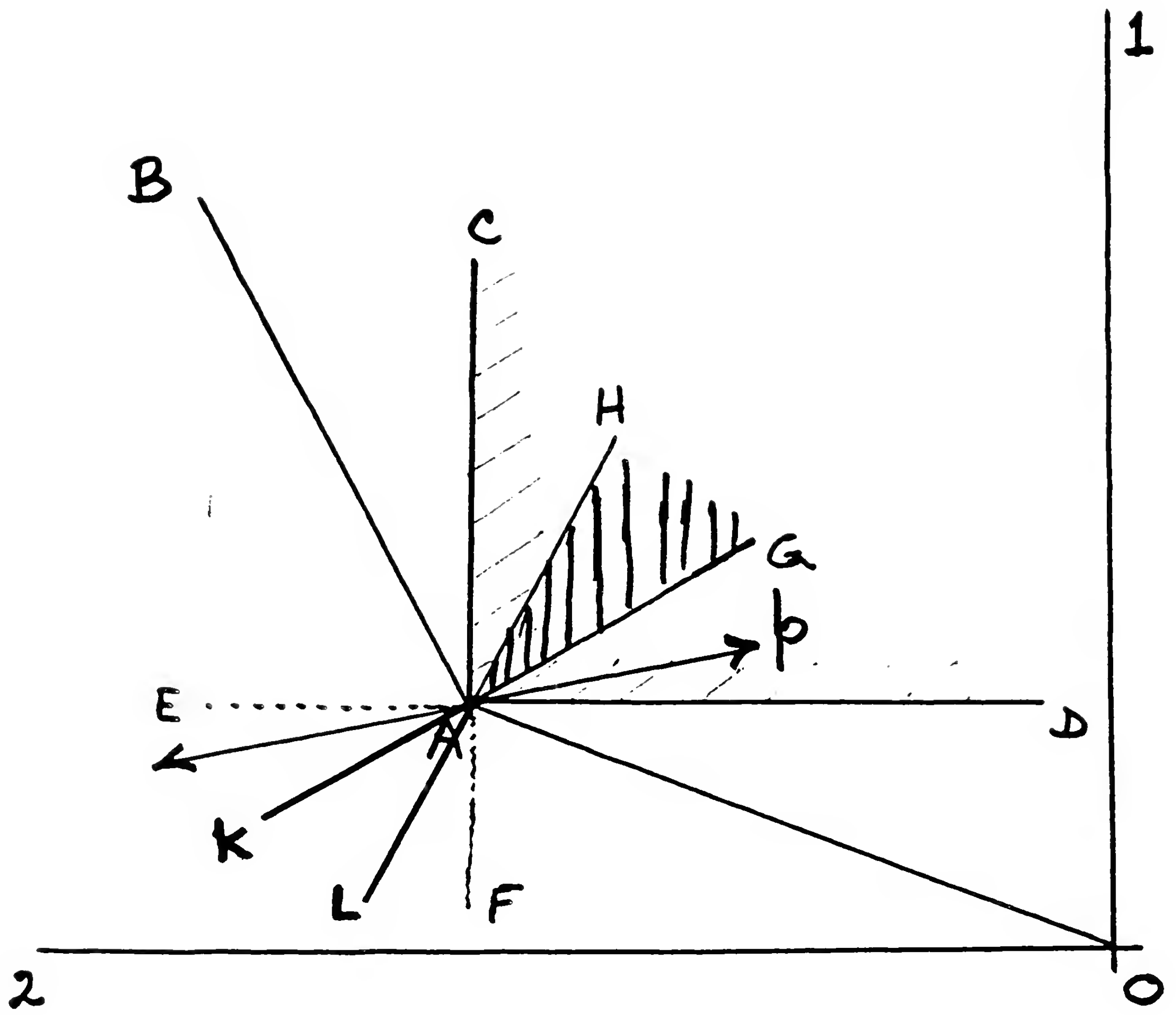

Figure 1.

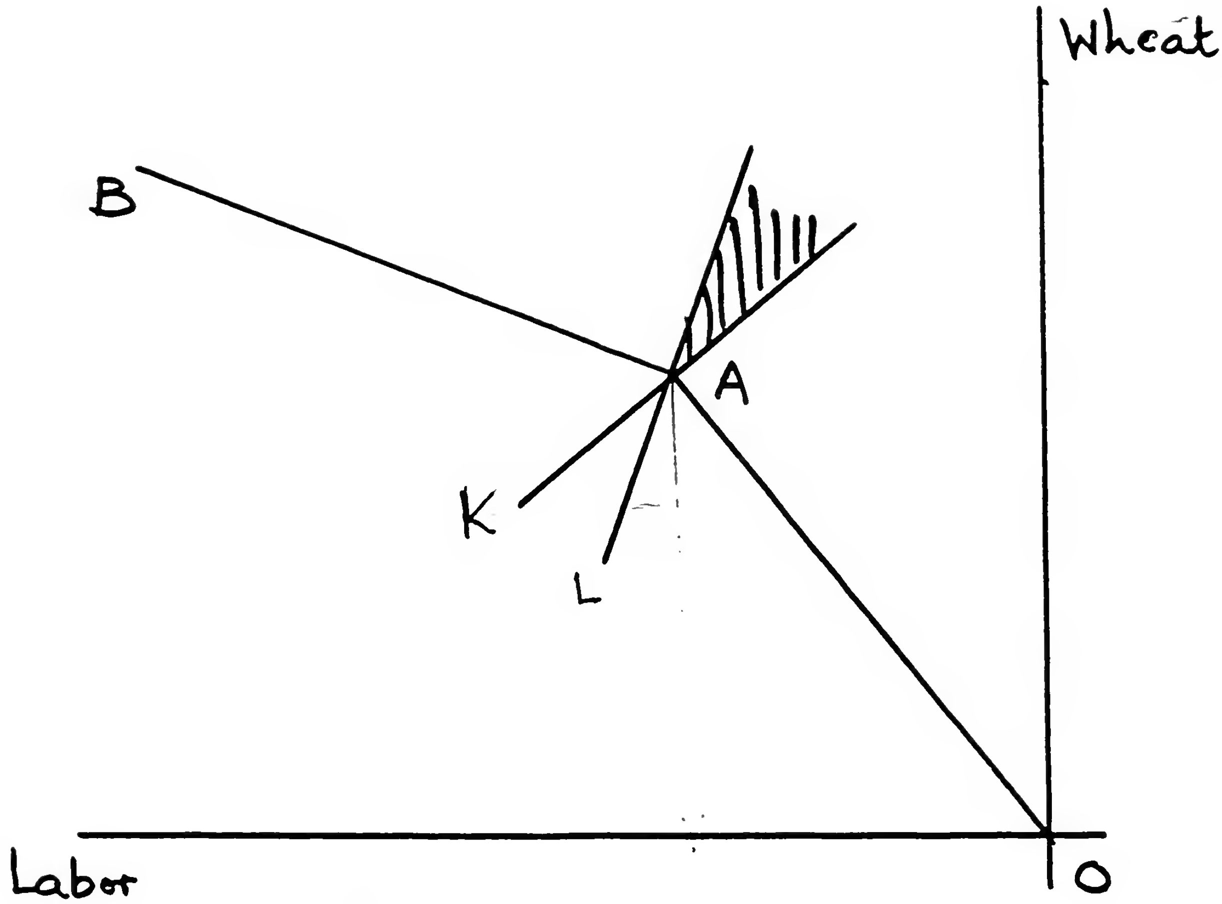

Figure 2a

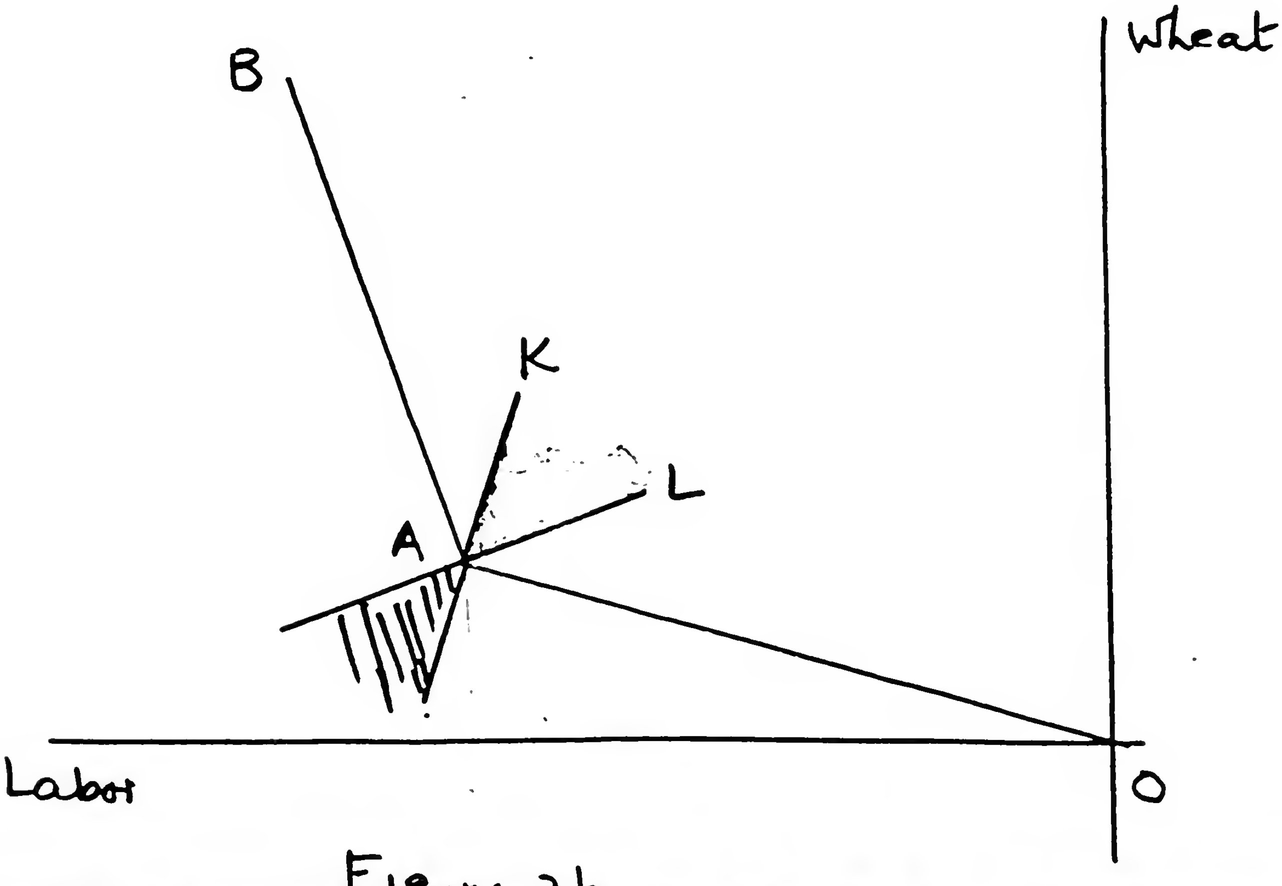

Figure 2b.

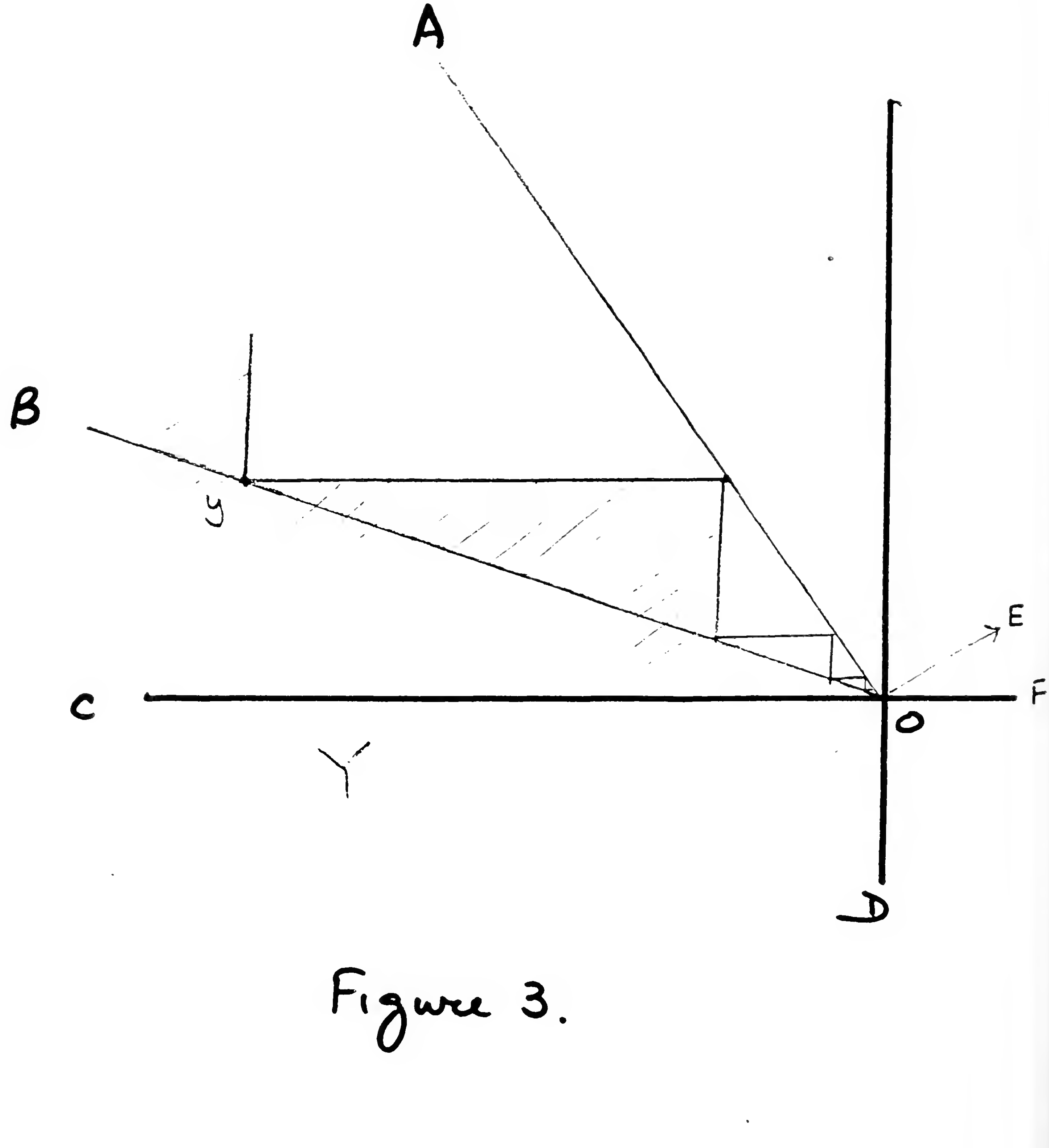

Figure 3.

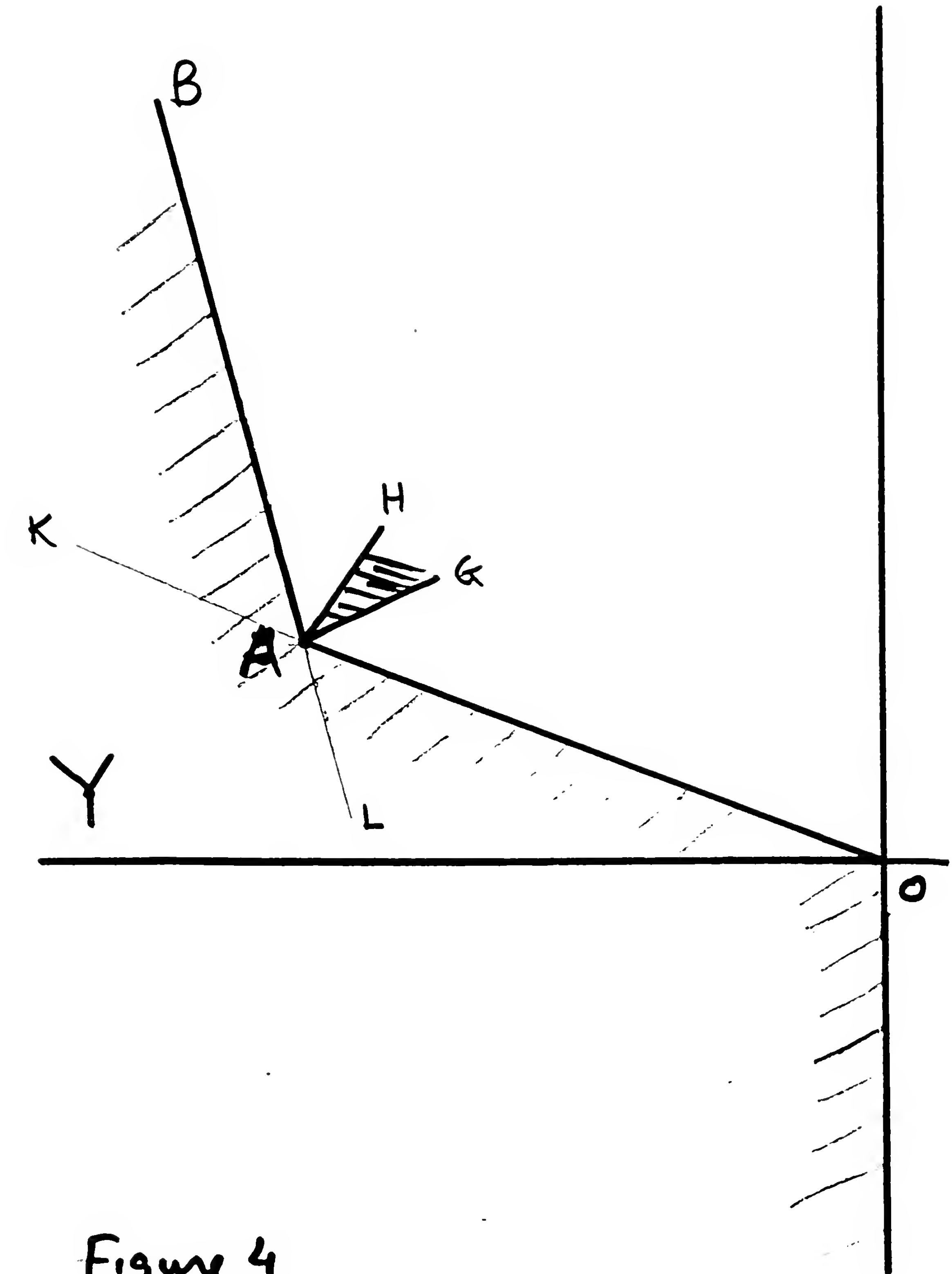

Figure 4

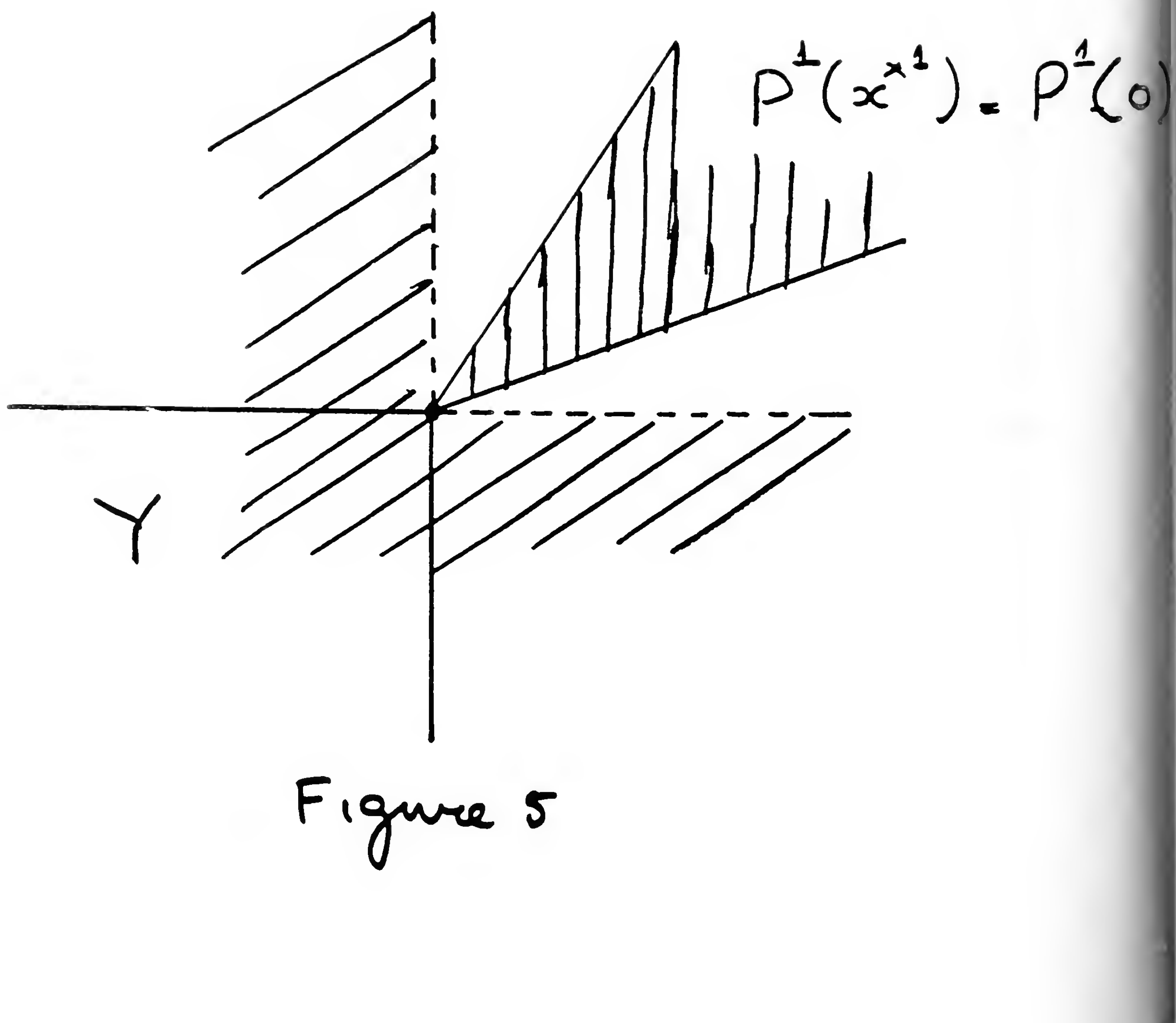

Figure 5